

# A STOCHASTIC HAMILTON-JACOBI EQUATION WITH INFINITE SPEED OF PROPAGATION

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ABSTRACT. We give an example of a stochastic Hamilton-Jacobi equation  $du = H(Du)d\xi$  which has an infinite speed of propagation as soon as the driving signal  $\xi$  is not of bounded variation.

## 1. INTRODUCTION

An important feature of (deterministic) Hamilton-Jacobi equations

$$(1.1) \quad \partial_t u = H(Du) \quad \text{on } (0, T) \times \mathbb{R}^N$$

is the so-called *finite speed of propagation* : assuming for instance that  $H : \mathbb{R}^N \rightarrow \mathbb{R}$  is  $C$ -Lipschitz, then if  $u^1$  and  $u^2$  are two (viscosity) solutions of (1.1), one has

$$(1.2) \quad u^1(0, \cdot) = u^2(0, \cdot) \text{ on } B(R) \Rightarrow \forall t \geq 0, u^1(t, \cdot) = u^2(t, \cdot) \text{ on } B(R - Ct)$$

where by  $B(R)$  we mean the ball of radius  $R$  centered at 0.

In this note, we are interested in Hamilton-Jacobi equations with rough time dependence of the form

$$(1.3) \quad \partial_t u = H(Du)\dot{\xi}(t) \quad \text{on } (0, T) \times \mathbb{R}^N,$$

where  $\xi$  is only assumed to be continuous. Of course, the above equation only makes classical (viscosity) sense for  $\xi$  in  $C^1$ , but Lions and Souganidis [2] have shown that if  $H$  is the difference of two convex functions, the solution map can be extended continuously (with respect to supremum norm) to any continuous  $\xi$ . (In typical applications, one wants to take  $\xi$  as the realization of a random process such as Brownian motion).

In fact, the Lions-Souganidis theory also gives the following result : if  $H = H_1 - H_2$  where  $H_1, H_2$  are convex,  $C$ -Lipschitz, with  $H_1(0) = H_2(0) = 0$ , then for any constant  $A$ ,

$$u(0, \cdot) \equiv A \text{ on } B(R) \Rightarrow u(t, \cdot) \equiv A \text{ on } B(R(t))$$

where  $R(t) = R - C(\max_{s \in [0, t]} \xi(s) - \min_{s \in [0, t]} \xi(s))$ .

However this does not imply a finite speed of propagation for (1.3) for arbitrary initial conditions, and a natural question (as mentioned in lecture notes by Souganidis [3]) is to know whether a property analogous to (1.2) holds in that case. The purpose

of this note is to show that in general it does not : we present an example of an  $H$  such that if the total variation of  $\xi$  on  $[0, T]$  is strictly greater than  $R$ , one may find initial conditions  $u_0^1, u_0^2$  which coincide on  $B(R)$ , but such that for the associated solutions  $u^1$  and  $u^2$ , one has  $u^1(T, 0) \neq u^2(T, 0)$ .

For instance, if  $\xi$  is a (realization of a) Brownian motion, then (almost surely), one may find initial conditions coinciding on balls of arbitrary large radii, but such that  $u^1(t, 0) \neq u^2(t, 0)$  for all  $t > 0$ .

It should be noted that the Hamiltonian  $H$  in our example is not convex (or concave). When  $H$  is convex, some of the oscillations of the path cancel out at the PDE level<sup>1</sup>, so that one cannot hope for simple bounds such as (2.2) below. Whether one has finite speed of propagation in this case remains an open question.

## 2. MAIN RESULT AND PROOF

We fix  $T > 0$  and denote  $\mathcal{P} = \{(t_0, \dots, t_n), 0 = t_0 \leq t_1 \leq \dots \leq t_n = T\}$  the set of partitions of  $[0, T]$ . Recall that the total variation of a continuous path  $\xi : [0, T] \rightarrow \mathbb{R}$  is defined by

$$V_{0,T}(\xi) = \sup_{(t_0, \dots, t_n) \in \mathcal{P}} \sum_{i=0}^{n-1} |\xi(t_{i+1}) - \xi(t_i)|.$$

Our main result is then :

**Theorem 1.** *Given  $\xi \in C([0, T])$ , let  $u : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be the viscosity solution of*

$$(2.1) \quad \partial_t u = (|\partial_x u| - |\partial_y u|) \dot{\xi}(t) \text{ on } (0, T) \times \mathbb{R}^2$$

*with initial condition*

$$u(0, x, y) = |x - y| + \Theta(x, y)$$

*where  $\Theta \geq 0$  is such that  $\Theta(x, y) \geq 1$  if  $\min x, y \geq R$ .*

*One then has the estimate*

$$(2.2) \quad u(T, 0, 0) \geq \left( \sup_{(t_0, \dots, t_n) \in \mathcal{P}} \frac{\sum_{j=0}^{n-1} |\xi(t_{j+1}) - \xi(t_j)|}{n} - \frac{R}{n} \right)_+ \wedge 1.$$

*In particular,  $u(T, 0, 0) > 0$  as soon as  $V_{0,T}(\xi) > R$ .*

Note that since  $|x - y|$  is a stationary solution of (2.1), the claims from the introduction about the speed of propagation follow.

The proof of Theorem 1 is based on the differential game associated to (2.1). Informally, the system is constituted of a pair  $(x, y)$  and the two players take turn controlling  $x$  or  $y$  depending on the sign of  $\dot{\xi}$ , with speed up to  $|\dot{\xi}|$ . The minimizing

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<sup>1</sup>for example : for  $\delta \geq 0$ ,  $S_H(\delta) \circ S_{-H}(\delta) \circ S_H(\delta) = S_H(\delta)$ , where  $S_H, S_{-H}$  are the semigroups associated to  $H, -H$ .

player wants  $x$  and  $y$  to be as close as possible to each other, while keeping them smaller than  $R$ . The idea is then that if the minimizing player keeps  $y$  and  $x$  stuck together, the maximizing player can lead  $x$  and  $y$  to be greater than  $R$  as long as  $V_{0,T}(\xi) > R$ .

*Proof of Theorem 1.* By approximation we can consider  $\xi \in C^1$ , and in fact we consider the backward equation :

$$(2.3) \quad \begin{cases} -\partial_t v &= (|\partial_x v| - |\partial_y v|) \dot{\xi}(t), \\ v(T, x, y) &= |x - y| + \Theta(x, y). \end{cases}$$

We then need a lower bound on  $v(0, 0, 0)$ . Note that

$$(|\partial_x v| - |\partial_y v|) \dot{\xi}(t) = \sup_{|a| \leq 1} \inf_{|b| \leq 1} \left\{ \dot{\xi}_+(t) (a \partial_x u + b \partial_y u) + \dot{\xi}_-(t) (a \partial_y u + b \partial_x u) \right\},$$

so that by classical results (e.g. [1]) one has the representation

$$(2.4) \quad v(0, 0, 0) = \sup_{\delta(\cdot) \in \Delta} \inf_{\beta \in \mathcal{U}} J(\delta(\beta), \beta),$$

where  $\mathcal{U}$  is the set of controls (measurable functions from  $[0, T]$  to  $[-1, 1]$ ) and  $\Delta$  the set of progressive strategies (i.e. maps  $\delta : \mathcal{U} \rightarrow \mathcal{U}$  such that if  $\beta = \beta'$  a.e. on  $[0, t]$ , then  $\delta(\beta)(t) = \delta(\beta')(t)$ ). Here for  $\alpha, \beta \in \mathcal{U}$ , the payoff is defined by

$$J(\alpha, \beta) = |x^{\alpha, \beta}(T) - y^{\alpha, \beta}(T)| + \Theta(x^{\alpha, \beta}(T), y^{\alpha, \beta}(T))$$

where

$$x^{\alpha, \beta}(0) = y^{\alpha, \beta}(0) = 0, \quad \dot{x}^{\alpha, \beta}(s) = \dot{\xi}_+(s)\alpha(s) + \dot{\xi}_-(s)\beta(s), \quad \dot{y}^{\alpha, \beta}(s) = \dot{\xi}_-(s)\alpha(s) + \dot{\xi}_+(s)\beta(s).$$

Assume  $v(0, 0, 0) < 1$  (otherwise there is nothing to prove) and fix  $\varepsilon \in (0, 1)$  such that  $v(0, 0, 0) < \varepsilon$ . Consider the strategy  $\delta^\varepsilon$  for the maximizing player defined as follow : for  $\beta \in \mathcal{U}$ , let

$$\tau_\varepsilon^\beta = \inf \{t \geq 0, \quad |x^{1, \beta}(t) - y^{1, \beta}(t)| \geq \varepsilon\},$$

and then

$$\delta^\varepsilon(\beta)(t) = \begin{cases} 1, & t < \tau_\varepsilon^\beta \\ \beta(t), & t \geq \tau_\varepsilon^\beta \end{cases}.$$

In other words, the maximizing player moves to the right at maximal speed, until the time when  $|x - y| = \varepsilon$ , at which point he moves in a way such that  $x$  and  $y$  stay at distance  $\varepsilon$ .

Now by (2.4), there exists  $\beta \in \mathcal{U}$  with  $J(\delta^\varepsilon(\beta), \beta) < \varepsilon$ . Clearly for the corresponding trajectories  $x(\cdot), y(\cdot)$ , this means that  $|x(T) - y(T)| < \varepsilon$ , and by definition of  $\alpha^\varepsilon$

this implies  $|x(t) - y(t)| \leq \varepsilon$  for  $t \in [0, T]$ . We now fix  $(t_0, \dots, t_n) \in \mathcal{P}$  and prove by induction that for  $i = 0, \dots, n$ ,

$$\min\{x(t_i), y(t_i)\} \geq \sum_{j=0}^{i-1} |\xi(t_{j+1}) - \xi(t_j)| - i\varepsilon.$$

Indeed, if it is true for some index  $i$ , then assuming that for instance  $\xi(t_{i+1}) - \xi(t_i) \geq 0$ , one has

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + \int_{t_i}^{t_{i+1}} \dot{\xi}_+(s) ds - \int_{t_i}^{t_{i+1}} \beta(s) \dot{\xi}_-(s) ds \\ &\geq x(t_i) + \xi(t_{i+1}) - \xi(t_i) \geq \sum_{j=0}^i |\xi(t_{j+1}) - \xi(t_j)| - i\varepsilon \end{aligned}$$

and since  $y(t_{i+1}) \geq x(t_{i+1}) - \varepsilon$ , one also has  $y(t_{i+1}) \geq \sum_{j=0}^i |\xi(t_{j+1}) - \xi(t_j)| - (i+1)\varepsilon$ . The case when  $\xi(t_{i+1}) - \xi(t_i) \leq 0$  is similar.

Since  $J(\alpha^\varepsilon(\beta), \beta) \leq 1$ , one must necessarily have  $\min\{x(T), y(T)\} \leq R$ , so that

$$\varepsilon \geq \frac{1}{n} \left( \sum_{j=0}^n |\xi(t_{j+1}) - \xi(t_j)| - R \right).$$

Letting  $\varepsilon \rightarrow v(0, 0, 0)$  and taking the supremum over  $\mathcal{P}$  on the r.h.s. we obtain (2.2).  $\square$

## REFERENCES

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